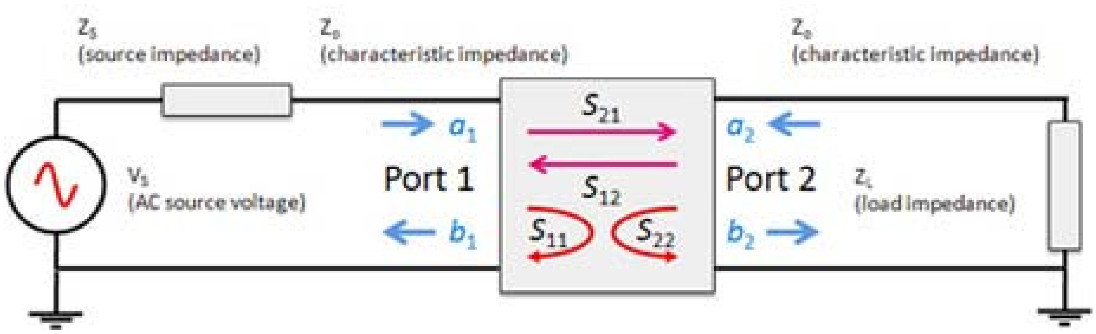
Network Techniques: Conversion between Filter Transfer Function and Filter Scattering (S-Matrix) Parameters

Syed Hisaam Hashim EE 172 – Dr. Ray Kwok 05/27/2011

Introduction

In low frequency systems, Ohm's Law is a suitable approximation for analysis of a circuit schematic and transfer function H(s) provides important details such as gain, stability, filter properties, and time-domain output information. On the contrary, at high frequencies (microwave and higher) Ohm's Law is not a suitable, and instead we must use the transmission line theory. The Scattering (S) Matrix parameters play a key role at higher frequencies by detailing a system's gain, return loss, voltage standing wave ratio (VSWR), reflection coefficient and amplifier stability (Wiki – Sparameters). When we are designing filters, commonly we are dealing with two-port networks where conveniently transfer function and S-Parameter analysis are both possible. It is therefore a very a useful tool to be able to take a filter's transfer function properties and demonstrate a relationship with its scattering parameters for high frequency information. In reality, there is a relationship between the two. S21 and H(s) have a causal relationship, and S11 is nearly the inverse of H(s) which can be derived in a lossless network using the lossless network property of S-Matrices. We can then use the reciprocity and symmetry properties of S-Matrices such that |S22|=|S11| and |S21|=|S12| where S22, S11, S21, and S12 may have different phase properties. Other papers related to this topic describe other relationships of the S-Parameters from transfer function qualities other than this method. A correlation between these two methods are not discussed in this paper but may be explored in future revisions on this topic.

Two-Port Networks for S-Matrix and Transfer Function H(s)



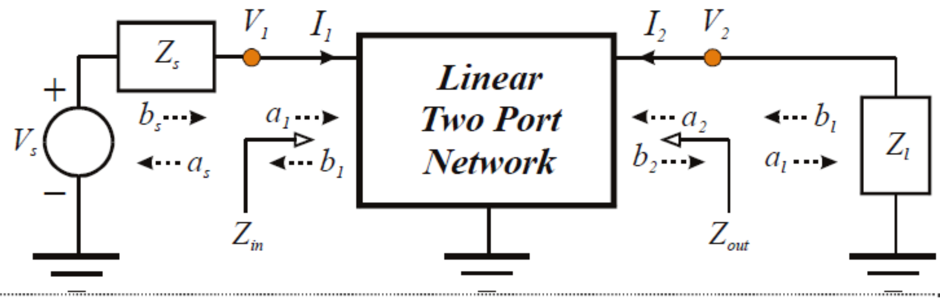


Fig. (11). Linear Network Terminated At Its Input And Output Ports In Generalized Load And Source Impedances. The Two Port Network Is Presumed To Have Measured Or Otherwise Known Scattering Parameters Referenced To A Characteristic Impedance Of R_o.

Figure 2: Schematic used for S-Matrix→H(s); Source: Course Notes #2, *Scattering Parameters: Concept, Theory, and Applications* by Dr. John Choma, Fall 2009, Section 3.1.2, [5]

The first step to derive a relationship between transfer functions and S-Matrices is to describe a two-port network in terms of impedances, voltages and power-waves necessary to derive S-Parameters and a general transfer function. Figure 1 shows a basic representation of what S11, S22, S21, and S12 are in terms of reflection and transmission of voltage-waves in a circuit.

S-Matrix Parameters to a Transfer Function

Reflection Coefficients

Looking at the movement of the 'a' and 'b' in Figure 2, we can derive some relationships for the reflection coefficients.

$$\Gamma_L = \frac{b_L}{a_L} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{a_2}{b_2} = \frac{1}{\Gamma_2} \qquad Z_{in} = \left(\frac{1+\Gamma_1}{1-\Gamma_1}\right) Z_0$$

$$\Gamma_S = \frac{b_S}{a_S} = \frac{Z_S - Z_0}{Z_S + Z_0} = \frac{a_1}{b_1} = \frac{1}{\Gamma_1} \qquad Z_{out} = \left(\frac{1+\Gamma_2}{1-\Gamma_2}\right) Z_0$$

We now begin by analyzing the circuit in terms of b2 and b1 in terms of a1 and a2 and find relationships for reflection coefficient 1 and 2.

$$b_2 = a_L = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 + S_{22}b_L = S_{21}a_1 + S_{22}\Gamma_L a_L$$

(1) $$b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1 + S_{12}b_L = S_{11}a_1 + S_{12}\Gamma_L a_L \qquad a_L(1 - S_{22}\Gamma_L) = S_{21}a_1$$

(2) $$a_L = \frac{S_{21}}{(1 - S_{22}\Gamma_L)} a_1 \qquad (2) \rightarrow (1) \qquad b_1 = S_{11}a_1 + S_{12}\Gamma_L \left(\frac{S_{21}}{1 - S_{22}\Gamma_L}\right) a_1$$

$$b_1 = \left[S_{11} + S_{12}\Gamma_L \left(\frac{S_{21}}{1 - S_{22}\Gamma_L}\right)\right] a_1$$

$$\Gamma_1 = \frac{b_1}{a_1} = S_{11} + \frac{\Gamma_L S_{21} S_{12}}{1 - \Gamma_L S_{22}}$$

$$\Gamma_2 = \frac{b_2}{a_2} = S_{22} + \frac{\Gamma_S S_{21} S_{12}}{1 - \Gamma_S S_{22}}$$

We now have all the relationships necessary to describe Zout and Zin.

Transfer Function H(s) = V2/Vs

We will now derive H(s) or the transfer function from the chain rule to get V2 in terms of Vs.

$$b_2 = \Gamma_2 a_2 = S_{21} a_1 + S_{22} a_2$$

$$a_2 (\Gamma_2 - S_{22}) = a_1 S_{21}$$

$$(3) \qquad \frac{a_2}{a_1} = \frac{S_{21}}{\Gamma_2 - S_{22}} = \frac{S_{21}}{\frac{1}{\Gamma_L} - S_{22}} = \frac{S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}$$

Chain Rule: $$tf_{source \to load} = \frac{V_L(s)}{V_S(s)} = \frac{V_2(s)}{V_S(s)} = \frac{V_2}{V_1} \cdot \frac{V_1}{V_S}$$

$$\frac{V_1}{V_S} = \frac{Z_{in}}{Z_{in} + Z_{out}} \qquad (VD) \qquad \frac{V_1}{V_S} = \frac{(1+\Gamma_1)(1-\Gamma_S)}{2(1-\Gamma_1\Gamma_S)}$$

$$\frac{V_2}{V_1} = \frac{a_2 + b_2}{a_1 + b_1} = \frac{a_2 + \Gamma_2 a_2}{a_1 + \Gamma_1 a_1} = \frac{a_2}{a_1}\left(\frac{1+\Gamma_2}{1+\Gamma_1}\right) \xrightarrow{(3)} \left(\frac{S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}\right)\left(\frac{1+\Gamma_2}{1+\Gamma_1}\right) \xrightarrow{\Gamma_L\Gamma_2 = 1} \left(\frac{S_{21}}{1 - S_{22}\Gamma_L}\right)\left(\frac{1+\Gamma_L}{1+\Gamma_1}\right)$$

$$H(s) = \frac{V_2(s)}{V_S(s)} = \frac{S_{21}}{2}\left[\frac{(1-\Gamma_S)(1-\Gamma_L)}{(1 - S_{11}\Gamma_S)(1 - S_{22}\Gamma_L) - S_{12}S_{21}(\Gamma_S\Gamma_L)}\right]$$

As we can see with this *complete* transfer function, we can see the relationship of H(s) in terms of S21 **only** by matching source and load impedances to the characteristic impedance of the line Z0. What this also demonstrates that the transfer function changes based on load and source impedances of the line, which makes sense at high frequencies because, in terms of the smith chart impedance, the total source impedance is moving away from the center (matched)

Transfer Function into S-Matrix Parameters

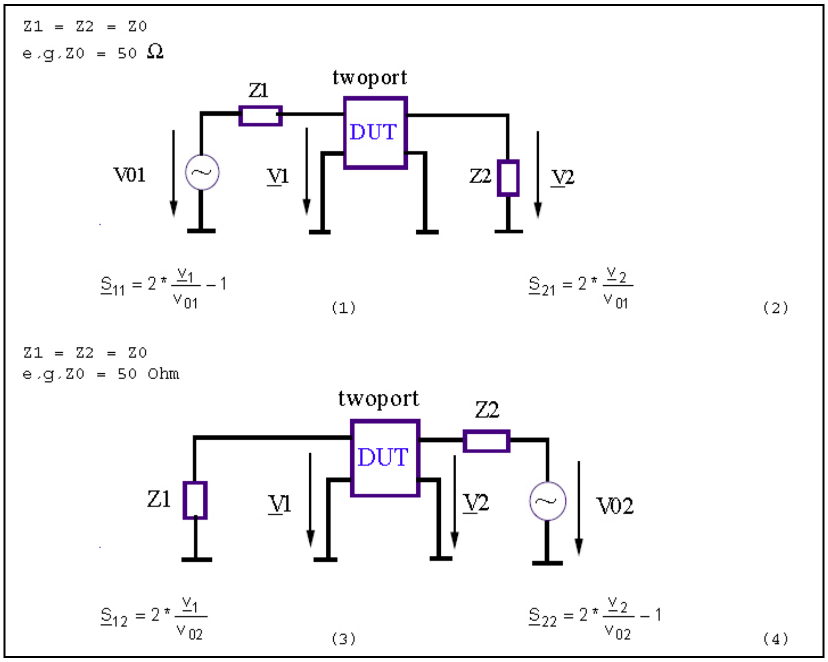


Figure 3: http://www.ece.unh.edu/courses/ece711/refrense_material/s_parameters/1SparBasics_1.pdf , p. 14, 2-Port S from different parts of circuit when matched

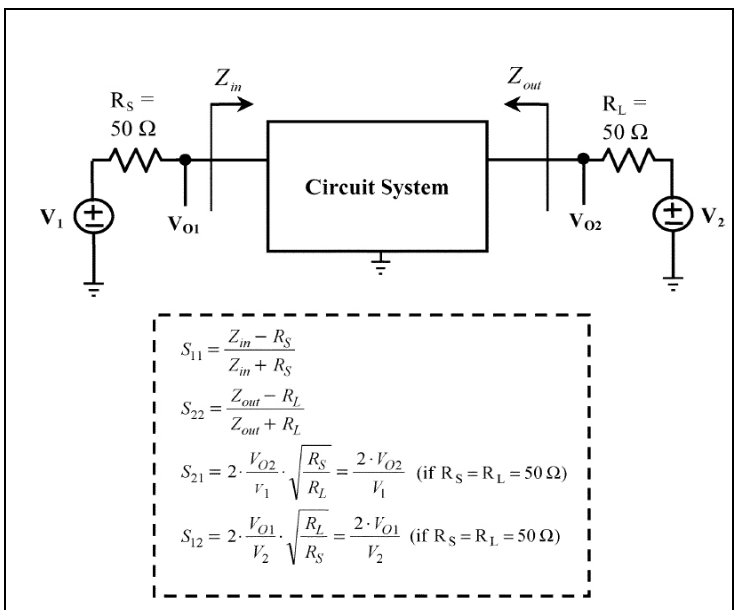


Fig. 1. Setup for the measurement/simulation of the S-parameters of a circuit system in the context of a 50-Ω measurement/simulation system.

Figure 4: *The determination of S-parameters from the poles of voltage-gain transfer function for RF IC design* (2005, Shey-Shi Lu, Yo-Sheng Lin, Hung-Wei Chiu, Yu-Chang Chen, Chin-Chun Meng)

The above two figures are two different methods to find the separate S-parameters from derivations presented in the previous section. However, I will offer another approach using S-Matrix properties:

Reciprocity

$$S_{ij} = S_{ji}$$
$$S_{12} = S_{21}(2\,port)$$

Symmetry

$$S_{ii} = S_{jj}$$
$$S_{11} = S_{22}$$

Lossless

$$\sum_{port-i} S_{i\cdot s} \bullet S_{i\cdot r} = 1 = \sum |S_{ij}|^2 = 1$$
$$|S_{11}|^2 + |S_{21}|^2 = 1$$
$$|S_{22}|^2 + |S_{12}|^2 = 1$$
$$|S_{21}| = \sqrt{1 - |S_{11}|^2}$$
$$|S_{12}| = \sqrt{1 - |S_{22}|^2}$$

Matched

$$S_{ii} = S_{jj} = 0$$
$$S_{11} = S_{22} = 0$$

Applying these properties we can derive S21, S12, S11, and S22.

$$\frac{b_2}{a_1} = S_{21}[/2] = G = tf = (A_0)\frac{Z(s)}{P(s)}$$

$$|S_{11}| = |S_{22}| \qquad |S_{12}| = |S_{21}|$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \left|\sqrt{1 - \left|A_0 \frac{Z(s)}{P(s)}\right|^2}\right| e^{j\theta_1} & \left|(A_0)\frac{Z(s)}{P(s)}\right| e^{j\phi} \\ (A_0)\frac{Z(s)}{P(s)} & \left|\sqrt{1 - \left|A_0 \frac{Z(s)}{P(s)}\right|^2}\right| e^{j\theta_2} \end{bmatrix}$$

Phase Difference (?)

$$S_{11} = |S_{11}| e^{j\theta_1}$$

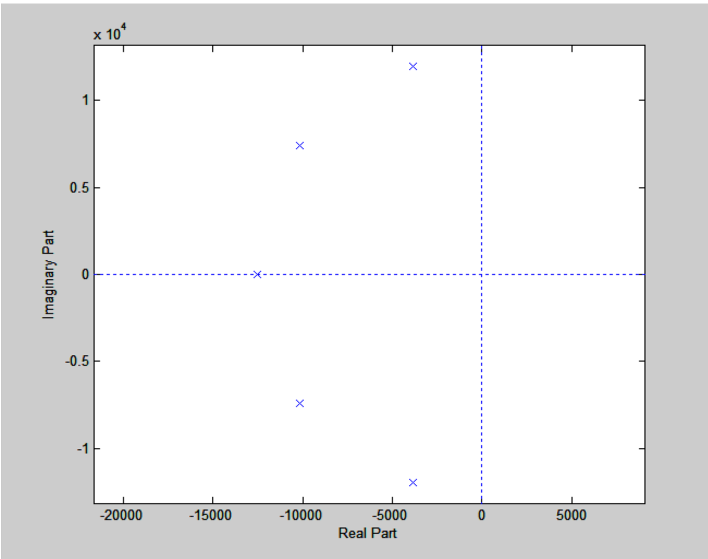
$$S_{22} = |S_{22}| e^{j\theta_2} \qquad \phi = \frac{\theta_1 + \theta_2}{2} \pm \frac{\pi}{2} \pm n\pi$$

$$S_{12} = |S_{12}| e^{j\phi}$$

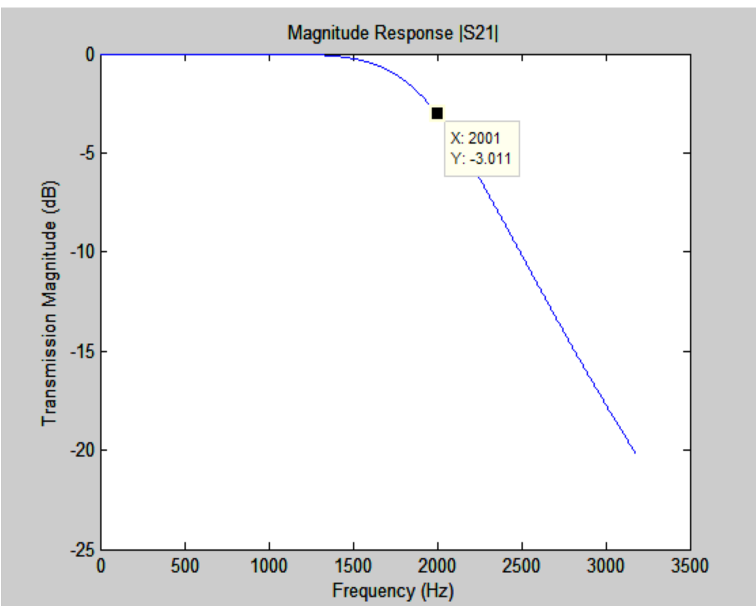
It should be noted that I assume S21 = H(s), but in our derivation, even when we are fully matched we get S21 = 2H(s)

Example

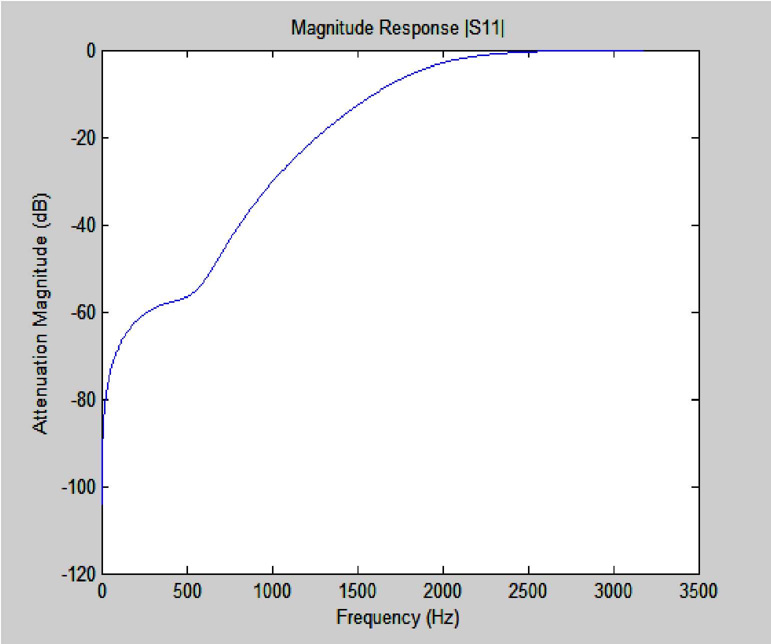
Pole-Zero info:



S21 (Magnitude Response)



S11 (Magnitude Response) using our equations.



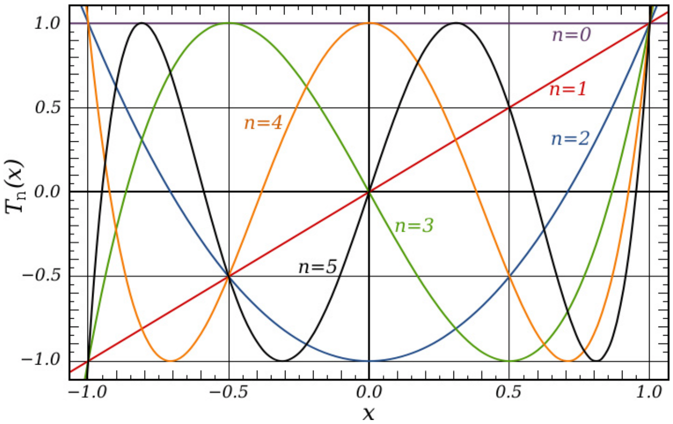
As we can see, S11 and S21 are nearly inverse from each other, which we do expect, because in real-life situations we see this reciprocal-like relationship between S21 and S11.

Filter Polynomials (Chebyshev and Butterworth)

Typically when we want to use these equations and transformations, we deal with popular filter polynomials, such as Chebyshev, Butterworth, and Elliptical. Due to the complexity with Elliptical filters, I will discuss Chebyshev and Butterworth Polynomials and the corresponding transfer functions.

Chebyshev Polynomial and Transfer Function

First Kind

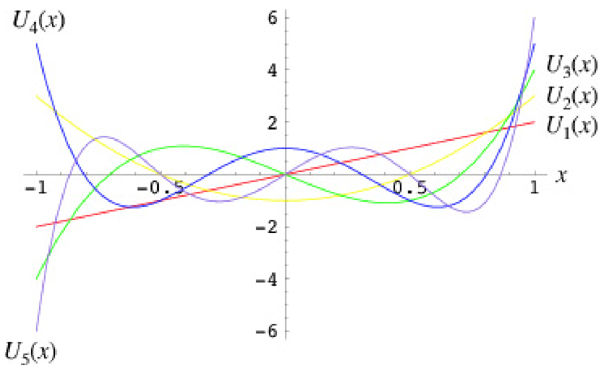


$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

$$T_n(x) = \frac{(x - \sqrt{x^2 - 1})^n + (x + \sqrt{x^2 - 1})^n}{2}$$

$$T_0\,(x) = 1$$
$$T_1\,(x) = x$$
$$T_2\,(x) = 2\,x^2 - 1$$
$$T_3\,(x) = 4\,x^3 - 3\,x$$
$$T_4\,(x) = 8\,x^4 - 8\,x^2 + 1$$
$$T_5\,(x) = 16\,x^5 - 20\,x^3 + 5\,x$$
$$T_6\,(x) = 32\,x^6 - 48\,x^4 + 18\,x^2 - 1.$$

Second Kind



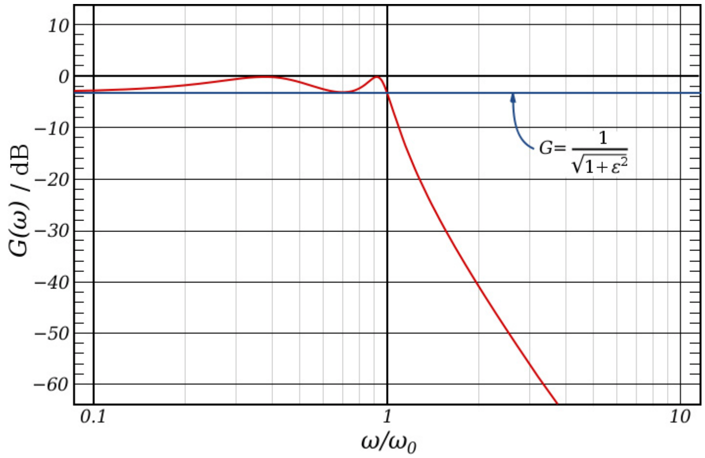
$$U_0(x) = 1$$
$$U_1(x) = 2x$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

$$U_n(x) = \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{2\sqrt{x^2 - 1}}$$

$$U_0\,(x) = 1$$
$$U_1\,(x) = 2\,x$$
$$U_2\,(x) = 4\,x^2 - 1$$
$$U_3\,(x) = 8\,x^3 - 4\,x$$
$$U_4\,(x) = 16\,x^4 - 12\,x^2 + 1$$
$$U_5\,(x) = 32\,x^5 - 32\,x^3 + 6\,x$$
$$U_6\,(x) = 64\,x^6 - 80\,x^4 + 24\,x^2 - 1.$$

Transfer Function (Chebyshev)

Type I

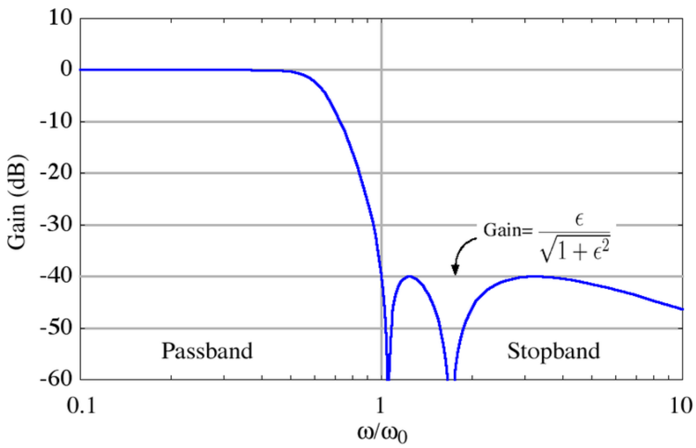


$$G_n(\omega) = |H_n(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 T_n^2\left(\frac{\omega}{\omega_0}\right)}}$$

Maxima and Minima: $G = 1/\sqrt{1+\varepsilon^2}$ (minima) and G=1 (maximum)

Ripple: Ripple in dB = $20\log_{10}\frac{1}{\sqrt{1+\varepsilon^2}}$

Type II



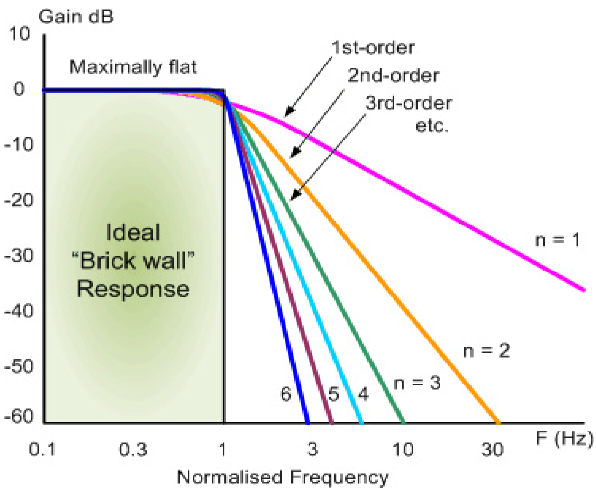
$$H(s) = \frac{1}{2^{n-1}\varepsilon}\prod_{m=1}^{n}\frac{1}{(s-s_{pm}^{-})}$$

$$s_{pm}^{\pm} = \pm \sinh\left(\frac{1}{n}\operatorname{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\sin(\theta_m)$$
$$+j\cosh\left(\frac{1}{n}\operatorname{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\cos(\theta_m)$$

where $m = 1, 2, \ldots, n$ and

$$\theta_m = \frac{\pi}{2}\frac{2m-1}{n}$$

Butterworth Polynomials and Transfer Function



$$B_n(s) = \prod_{k=1}^{\frac{n}{2}}\left[s^2 - 2s\cos\left(\frac{2k+n-1}{2n}\,\pi\right) + 1\right] \text{for n even}$$
$$B_n(s) = (s+1)\prod_{k=1}^{\frac{n-1}{2}}\left[s^2 - 2s\cos\left(\frac{2k+n-1}{2n}\,\pi\right) + 1\right] \text{for n odd}$$

n	Normalised Denominator Polynomials in Factored Form
1	$(1+s)$
2	$(1+1.414s+s^2)$
3	$(1+s)(1+s+s^2)$
4	$(1+0.765s+s^2)(1+1.848s+s^2)$
5	$(1+s)(1+0.618s+s^2)(1+1.618s+s^2)$
6	$(1+0.518s+s^2)(1+1.414s+s^2)(1+1.932s+s^2)$
7	$(1+s)(1+0.445s+s^2)(1+1.247s+s^2)(1+1.802s+s^2)$
8	$(1+0.390s+s^2)(1+1.111s+s^2)(1+1.663s+s^2)(1+1.962s+s^2)$
9	$(1+s)(1+0.347s+s^2)(1+s+s^2)(1+1.532s+s^2)(1+1.879s+s^2)$
10	$(1+0.313s+s^2)(1+0.908s+s^2)(1+1.414s+s^2)(1+1.782s+s^2)(1+1.975s+s^2)$

Transfer Function (Butterworth)

$$s_k = \omega_c e^{\frac{j(2k+n-1)\pi}{2n}} \qquad \mathrm{k} = 1, 2, 3, \ldots, \mathrm{n}$$

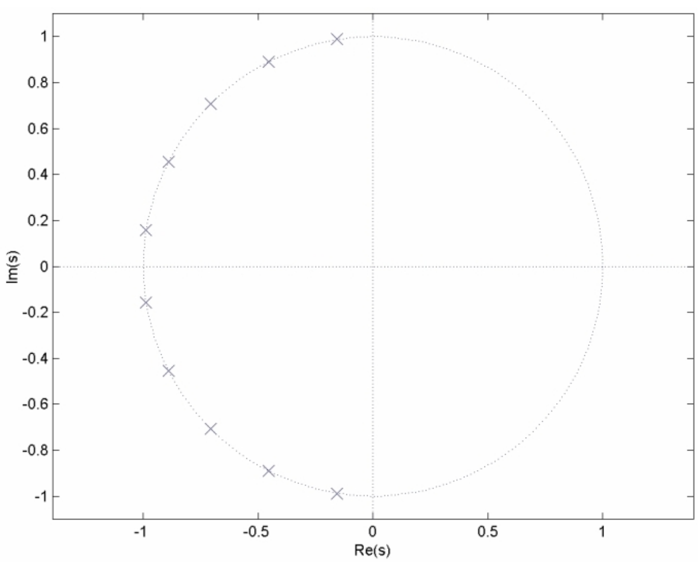
The transfer function may be written in terms of these poles as;

$$H(s) = \frac{G_0}{\prod_{k=1}^{n}(s - s_k)/\omega_c}$$

Pole-Zero Chebyshev and Butterworth

Normalized Butterworth:

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}}$$

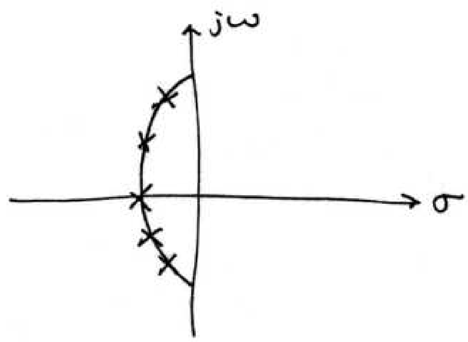


Notice that the Poles lie on the unit-circle in pole-zero plot [18]

Normalized Chebyshev (Type 1)

$$|H(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 {C_n}^2(\omega)}}$$

$C_n(\omega) = \cos\left(n \arccos(\omega)\right)$ is the Chebyshev Polynomial (Tn(x))



The poles lie on the ellipse on the left-hand side ONLY compared to the Butterworth which is on the unit-circle. [19]

Filter Design: Frequency Shifting

In Chapter 8 of *Microwave Engineering (3ED)* by Pozar, the author defines something called the power loss ratio or P_LR.

$$P_{LR} = \frac{1}{1-\left|\Gamma(\omega)\right|^2}$$

$$IL = 10\log(P_{LR})$$

$$IL = 10\log\left(\frac{1}{\left|S_{12}\right|^2}\right) \text{ at matched loads}$$

High-Pass

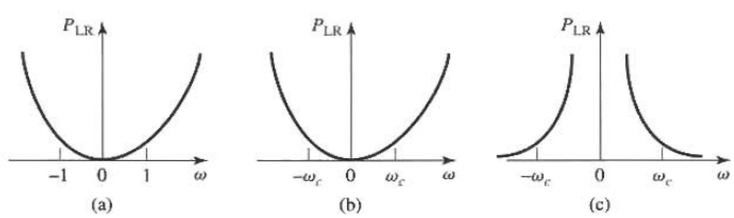


FIGURE 8.28 Frequency scaling for low-pass filters and transformation to a high-pass response. (a) Low-pass filter prototype response for $\omega_c = 1$. (b) Frequency scaling for low-pass response. (c) Transformation to high-pass response.

Figure 5: Ch. 8. Microwave Engineering, Pozar, on different filters by manipulating Lowpass

$$\omega_{LP} \leftarrow \frac{-\omega_c}{\omega}$$

Band-Pass

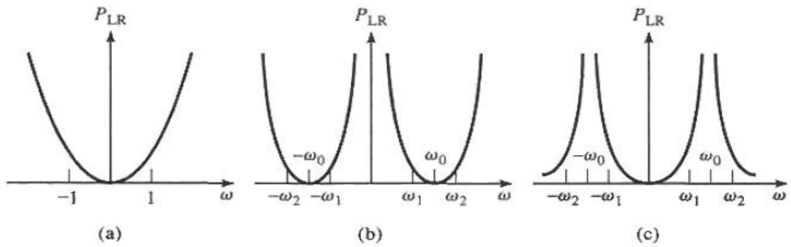


FIGURE 8.31 Bandpass and bandstop frequency transformations. (a) Low-pass filter prototype response for $\omega_c = 1$. (b) Transformation to bandpass response. (c) Transformation to bandstop response.

Figure 6: Ch. 8. Microwave Engineering, Pozar, on different filters by manipulating Lowpass

$$\omega_{LP} \leftarrow \frac{\omega_0}{\omega_2 - \omega_1}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{1}{\nabla}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \qquad where \qquad \nabla = \frac{\omega_2 - \omega_1}{\omega_0}$$

$$choose \qquad \omega_0 = \sqrt{\omega_2 \omega_1}$$

Conclusion and Improvements

As we have discussed in the paper, 2-port S-Matrix parameters and a 2-port transfer function are related to each other. However, there are several problems. Especially for S-Matrix parameters in terms of H(s), different papers on this topic offer a different interpretation on the derivation and a causal relationship between both the paper’s derivation and the scholarly article derivation is difficult to correlate. This was actually a major problem in the project because I wasn’t too sure how to support my work with existing theories. However, graphs-wise my method seems to work, mostly in terms of magnitude response of the parameters (scattering). I did not have the time to research or discuss group-delay which is essentially the phase response of the network at high-frequencies, which may have been important. This would be something to work on for a future improvement. Other future improvements include adding good examples starting with a filter polynomial’s pole-zero specifications and convert them into S-Parameters, as well as an example of an S-Matrix converted into a Transfer Function.

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