Chapter 2

Vector Algebra Review

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Vector products

(scalar) (scalar) = scalar, (a)(b) = ab
e.g. 2(4 kg) = 8 kg

(scalar) (vector) = vector, $k(\vec{A}) = k\vec{A}$
e.g. $5(2\hat{x} + 4\hat{y}) = 10\hat{x} + 20\hat{y}$

(vector) times (vector) = ?
can be scalar (scalar product, or dot product) $\vec{\mathrm{A}} \cdot \vec{\mathrm{B}}$
or
vector (vector product, or cross product) $\vec{\mathrm{A}} \times \vec{\mathrm{B}}$

Can you add a vector to a scalar?

The scalar product (dot product)

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\hat{x}\cdot\hat{x} = 1 = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z}$$

$$\hat{x}\cdot\hat{y} = 0 = \hat{x}\cdot\hat{z} = \hat{y}\cdot\hat{z}$$

$$\vec{A}\cdot\vec{B} = \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)\cdot\left(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}\right)$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

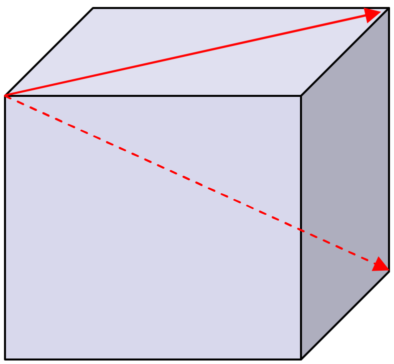
e.g.

$$\vec{A} = 2\hat{x} - 2\hat{y}$$

$$\vec{B} = 3\hat{x} + \hat{y}$$

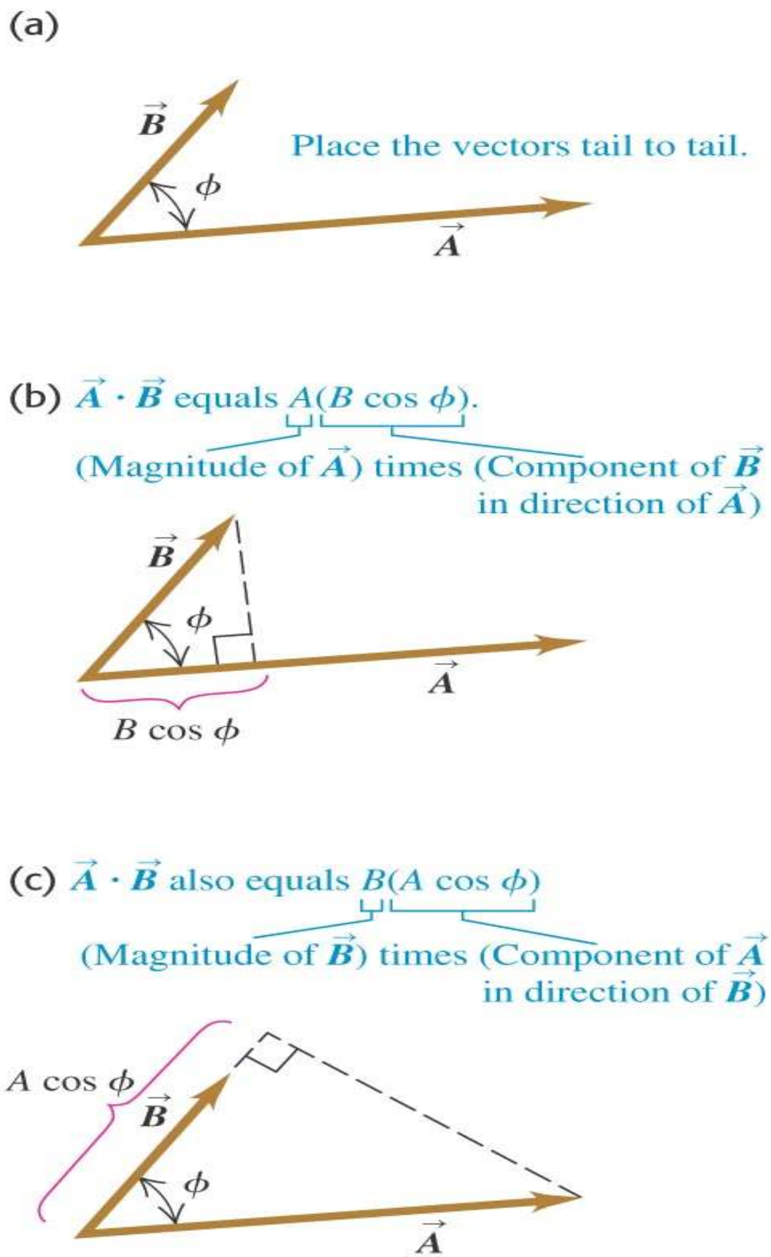
What is **A · B** ?

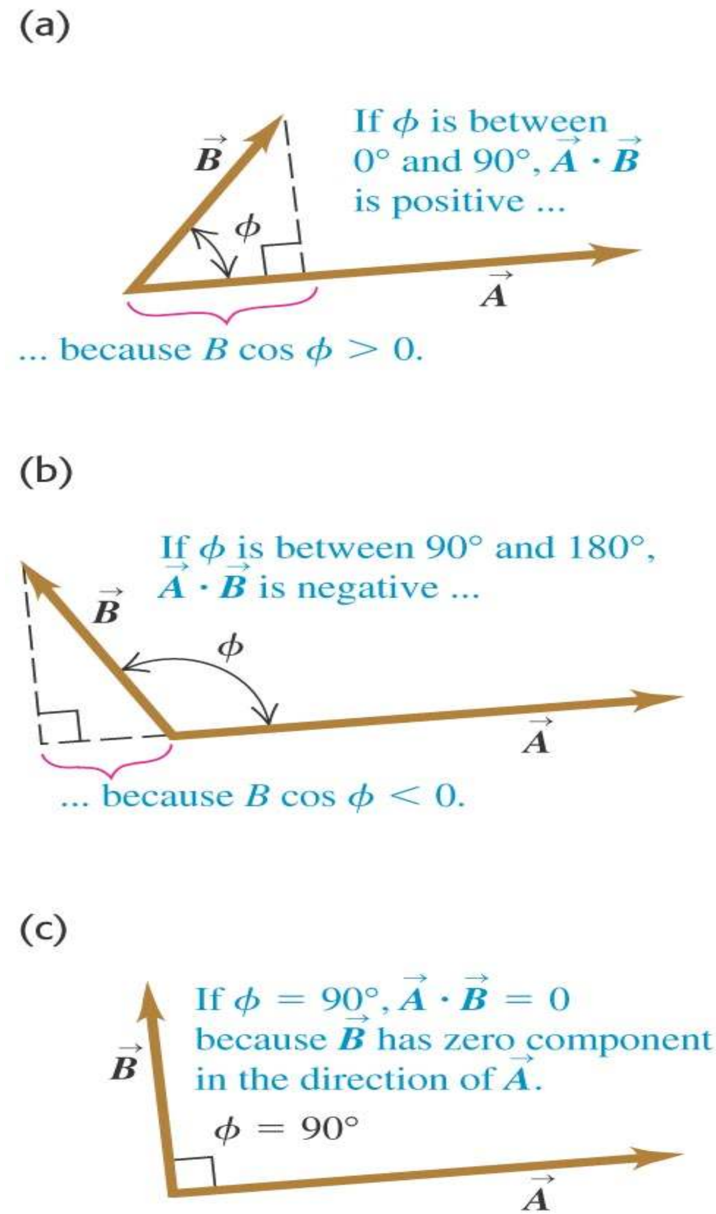
e.g.



What's the angle between these 2 arrows?

Interpretation - projection

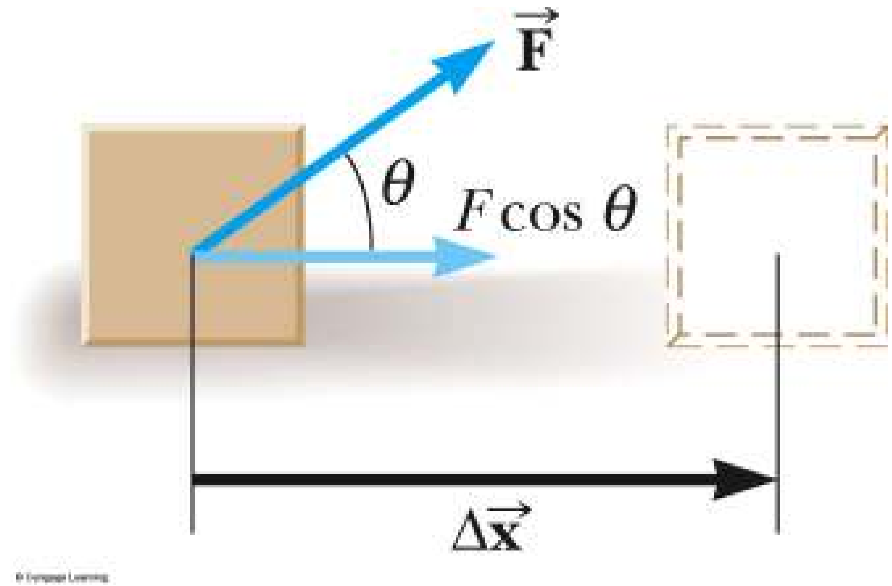




Example - Work

$$W = \vec{F} \cdot \Delta\vec{x}$$

(Projection)



So, is the uplifting force doing anything at all??

The vector product (cross product)

$$\left|\vec{A}\times\vec{B}\right| = AB\sin\theta$$

$$\hat{x}\times\hat{x} = 0 = \hat{y}\times\hat{y} = \hat{z}\times\hat{z}$$

$$\hat{x}\times\hat{y} = \hat{z}$$

$$\hat{y}\times\hat{z} = \hat{x}$$

$$\hat{z}\times\hat{x} = \hat{y}$$

right-hand coordinate
cyclic permutation

$$\hat{y}\times\hat{x} = -\hat{z}$$

$$\hat{z}\times\hat{y} = -\hat{x}$$

$$\hat{x}\times\hat{z} = -\hat{y}$$

$$\vec{A}\times\vec{B} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

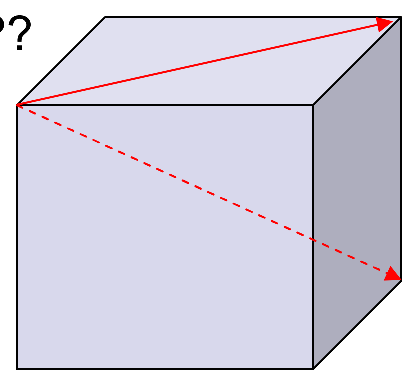
e.g.

$$\vec{A} = 2\hat{x} - 2\hat{y}$$

$$\vec{B} = 3\hat{x} + \hat{y}$$

What is **A** x **B** ?

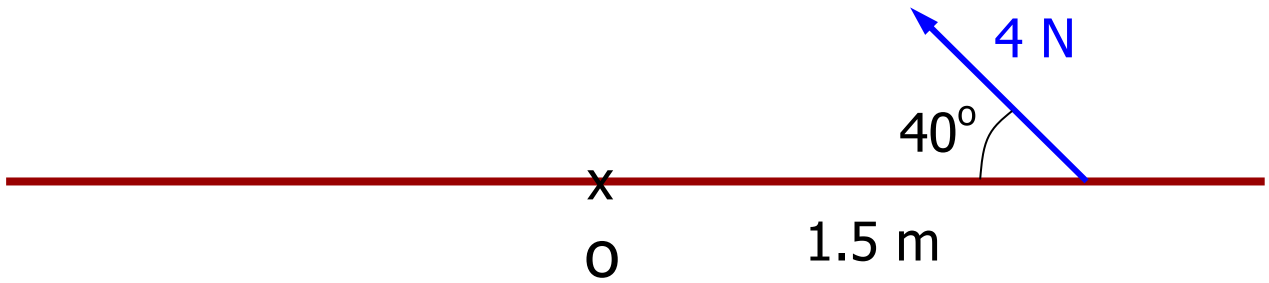
e.g.??



What's the angle between these 2 arrows?

Example – calculate torque

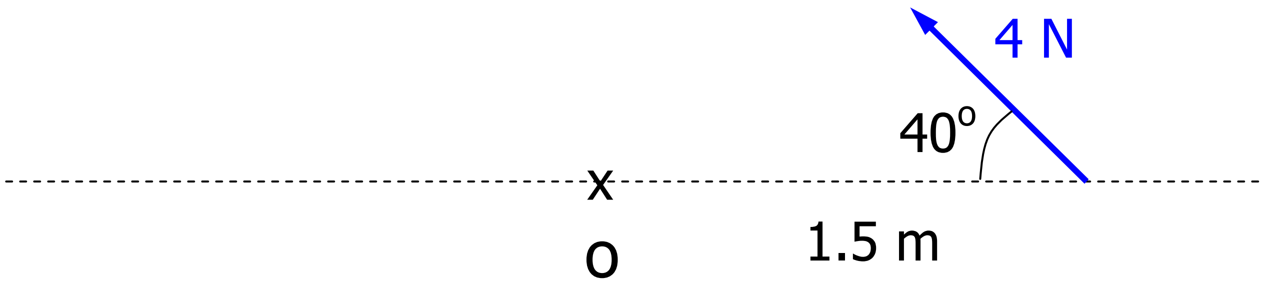
$$\vec{\tau} = \vec{r} \times \vec{F}$$

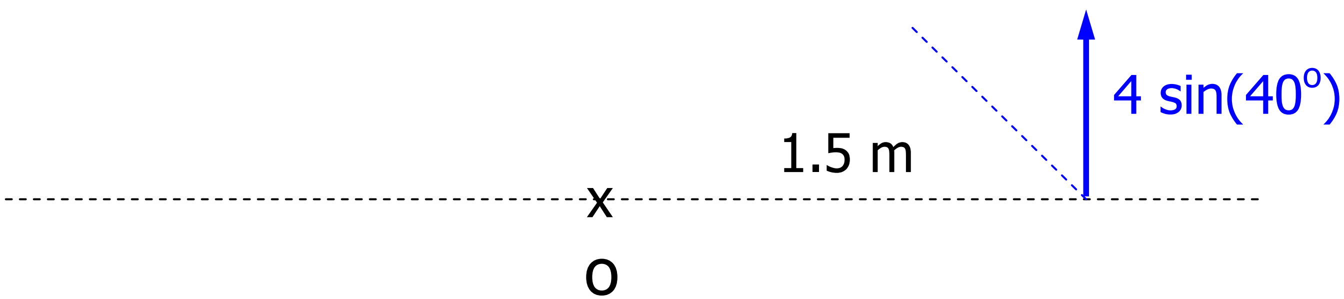


choose "+" = counter-clockwise

$\Sigma\tau_o = (1.5)(4)\sin(40^o)$
$= 3.86$ N-m (counter-clockwise)

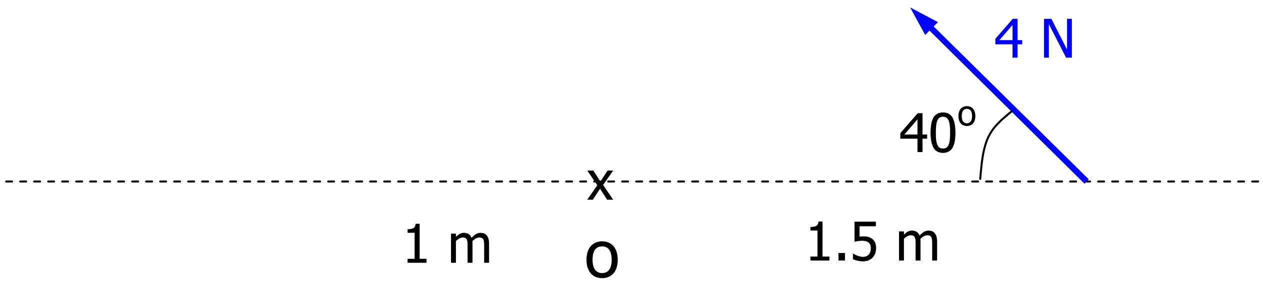
Example – perpendicular F

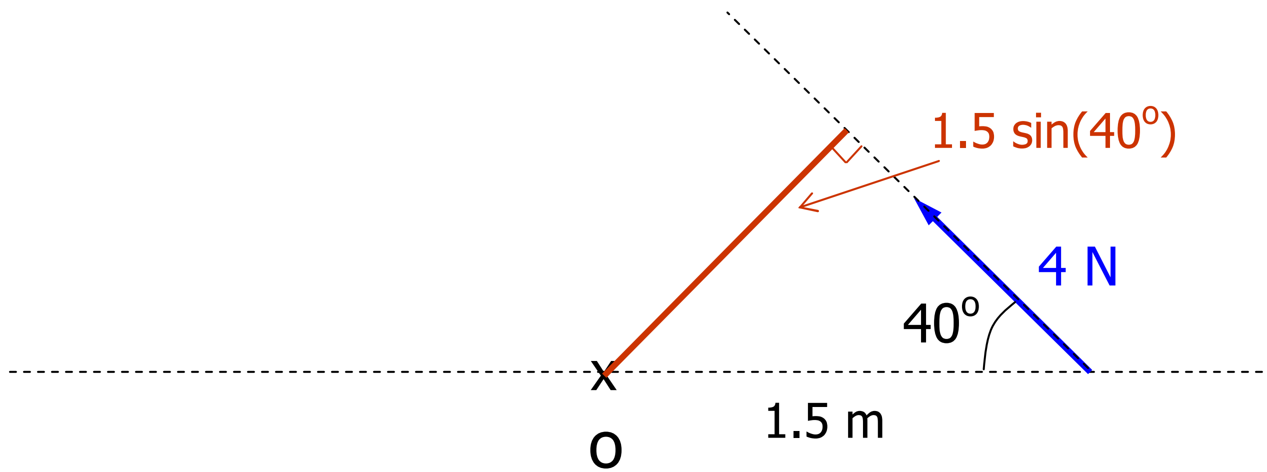




$$\Sigma\tau_o = (1.5)[4\sin(40^o)]$$
$$= 3.86 \text{ N-m} \quad \text{(counter-clockwise)}$$

Example – moment arm





$$\Sigma\tau_o = (4)[1.5\sin(40^o)]$$
$$= 3.86 \text{ N-m} \quad \text{(counter-clockwise)}$$

Exercise - 1

$$\vec{A} = \hat{x} - 4\hat{z}$$

$$\vec{B} = 2\hat{x} + \hat{y} + \hat{z}$$

Find:

(a) $\vec{A} + \vec{B}$

(b) $\vec{B} - 2\vec{A}$

(c) $\vec{A} \cdot \vec{B}$

(d) $\vec{A} \times \vec{B}$

(e) $\vec{A} \times \vec{A}$

(f) $\vec{B} \cdot \vec{B}$

(g) Angle between A and B

(h) Find a vector that is perpendicular to A and B ?

Triple vector product

scalar

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

vector

$$(\vec{A}\times\vec{B})\times\vec{C}=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{B}\cdot\vec{C})\vec{A}\neq\vec{A}\times(\vec{B}\times\vec{C})$$

(homework)

OCC Orthogonal Curvilinear Coordinates

Orthogonal $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

Curvilinear – coordinate surfaces can be curved

Cartesian, Cylindrical & Spherical coordinates

- Transformation between coordinates
- Line, Area & Volume integral in each coordinate

Rectangular ⇔ Polar (2D)

- Polar to rectangular
  - $x = r \cos \theta$
  - $y = r \sin \theta$
- Rectangular to polar
  - $r^2 = x^2 + y^2$ (Pythagorean theorem)
  - $\tan \theta = y/x$ (be certain which angle is θ)

Transformation

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r \cos\phi$ $y = r \sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r \cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta \cos\phi + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta \cos\phi + \hat{\mathbf{y}} \cos\theta \sin\phi - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$	$A_R = A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = R \sin\theta \cos\phi$ $y = R \sin\theta \sin\phi$ $z = R \cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta \cos\phi + \hat{\boldsymbol{\theta}} \cos\theta \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}} \sin\theta \sin\phi + \hat{\boldsymbol{\theta}} \cos\theta \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_x = A_R \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$ $A_y = A_R \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}} \sin\theta + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cos\theta - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r \sin\theta + A_z \cos\theta$ $A_\theta = A_r \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R \sin\theta$ $\phi = \phi$ $z = R \cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_r = A_R \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$

Vector operations

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\|A\| =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $d\mathcal{V} =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Homework

Ch.2 - 3, 5, 10, 12, 13, 15, 20, 23, 26, 30,

and prove eqn-2.33 $(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}$

Also prove that $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$