Maxwell Equations

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•reference:
Electromagnetic Fields and Waves, Lorrain & Corson (Freeman)
Introduction to Electrodynamics, D.J. Griffiths (Prentice Hall)
Fundamentals of Engineering Electromagnetics, D.K. Cheng (Addison Wesley)

Electrodynamics

$$\left\{\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho_t}{\varepsilon_o} \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{B} &= \mu_o \vec{J}_t \end{aligned}\right\}$$

Gauss's Law

Faraday's Law

No magnetic charge

Ampere's Law

Inconsistence ?

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \cdot \vec{B}\right)$$

div. of curl = 0
div. of B = 0
ok

$$\nabla \times \vec{B} = \mu_o \vec{J}_t$$

$$\nabla \cdot \nabla \times \vec{B} = \mu_o \left(\nabla \cdot \vec{J}_t\right)$$

$$\nabla \cdot \vec{J}_t = 0$$

??? why ?

Continuity Equation

$$\frac{d}{dt}\rho(\vec{r}(t),t) = \frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial\vec{r}}\cdot\frac{d\vec{r}}{dt}$$

notation (?)

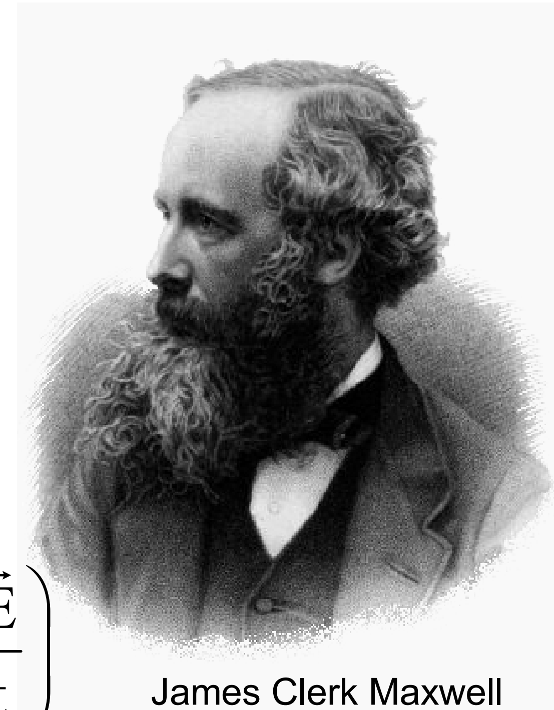
$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial\vec{r}}\cdot\vec{v} = \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\vec{r}}\cdot(\rho\vec{v}) = \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J}$$

$$0 = \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J}$$

conservation of "matter": $d\rho/dt = 0$

Here, conservation of charges….free, bound, total.

Maxwell's work



James Clerk Maxwell
(1831 - 1879)

Gauss's Law

$$\frac{\partial \rho_t}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon_o \nabla \cdot \vec{E}\right) = \nabla \cdot \left(\varepsilon_o \frac{\partial \vec{E}}{\partial t}\right)$$

Continuity Eqn

$$0 = \frac{\partial \rho_t}{\partial t} + \nabla \cdot \vec{J}_t = \nabla \cdot \left(\vec{J}_t + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right)$$

Faraday

$$\nabla \times \vec{E} = 0 \quad \Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell

$$\nabla \times \vec{B} = \mu_o \vec{J}_t \quad \Rightarrow \nabla \times \vec{B} = \mu_o \left(\vec{J}_t + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right)$$

~displacement current density

such that

$$\nabla \cdot \nabla \times \vec{B} = \mu_o \nabla \cdot \left(\vec{J}_t + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right) = 0$$

Maxwell Equations

$$\nabla \cdot \vec{E} = \frac{\rho_t}{\varepsilon_o}$$ Gauss's Law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Faraday's Law

$$\nabla \cdot \vec{B} = 0$$ No magnetic charge

$$\nabla \times \vec{B} = \mu_o \left(\vec{J}_t + \varepsilon_o \frac{\partial \vec{E}}{\partial t} \right)$$ Modified Ampere's Law (Maxwell)

Together with Lorentz Force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

describe all classical electrodynamics in free space

But, what are ρ and J?

In dielectrics

linear, homogeneous, isotropic

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_o\varepsilon_r\vec{E} = \varepsilon_o\vec{E} + \vec{P}$$

$$\nabla\cdot\vec{D} = \varepsilon_o\nabla\cdot\vec{E} + \nabla\cdot\vec{P}$$

$$\rho_f = \rho_t - \rho_b$$

bound charge $\nabla\cdot\vec{P} = -\rho_b$

$$\rho_t \equiv \rho_f + \rho_b = \frac{\rho_f}{\varepsilon_r} \le \rho_f$$

$$\rho_b \le 0$$

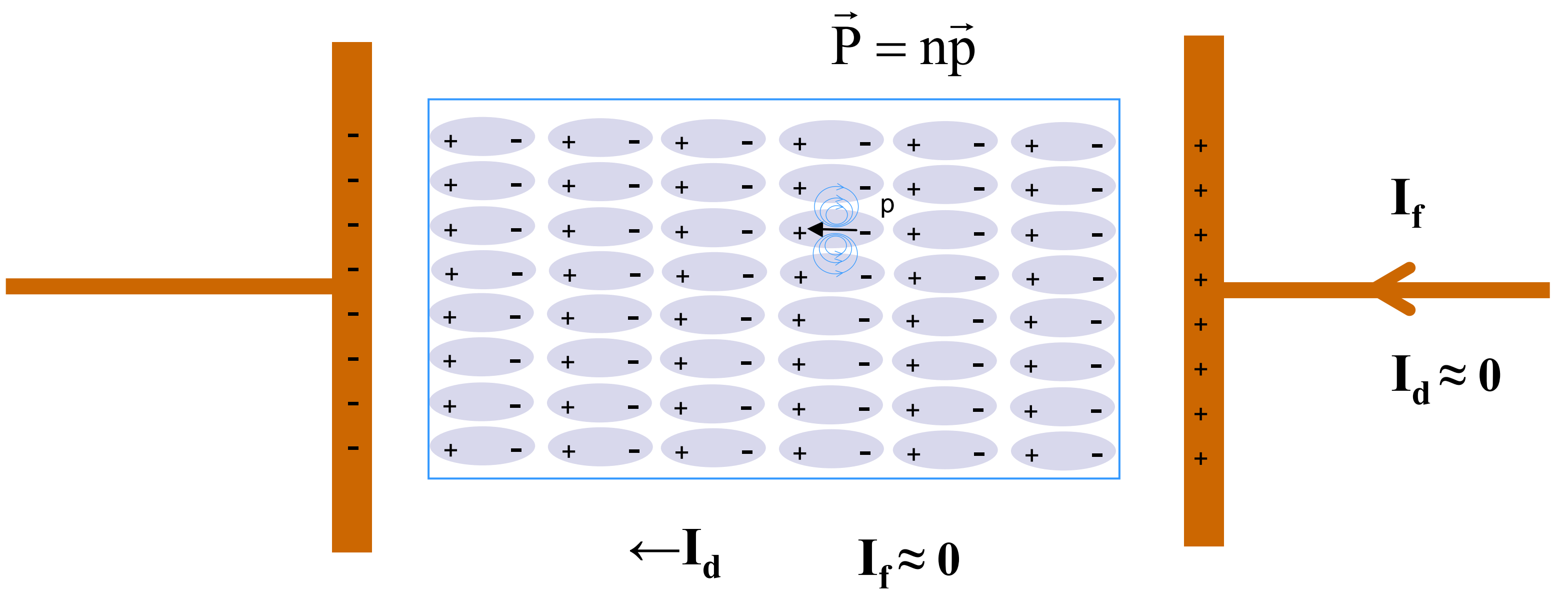
$$\nabla\cdot\vec{J}_t = \nabla\cdot\left(\vec{J}_f + \vec{J}_b\right) = \nabla\cdot\vec{J}_f - \frac{\partial\rho_b}{\partial t} = \nabla\cdot\vec{J}_f + \frac{\partial}{\partial t}(\nabla\cdot\vec{P})$$

continuity

$$\vec{J}_t = \vec{J}_f + \frac{\partial\vec{P}}{\partial t}$$

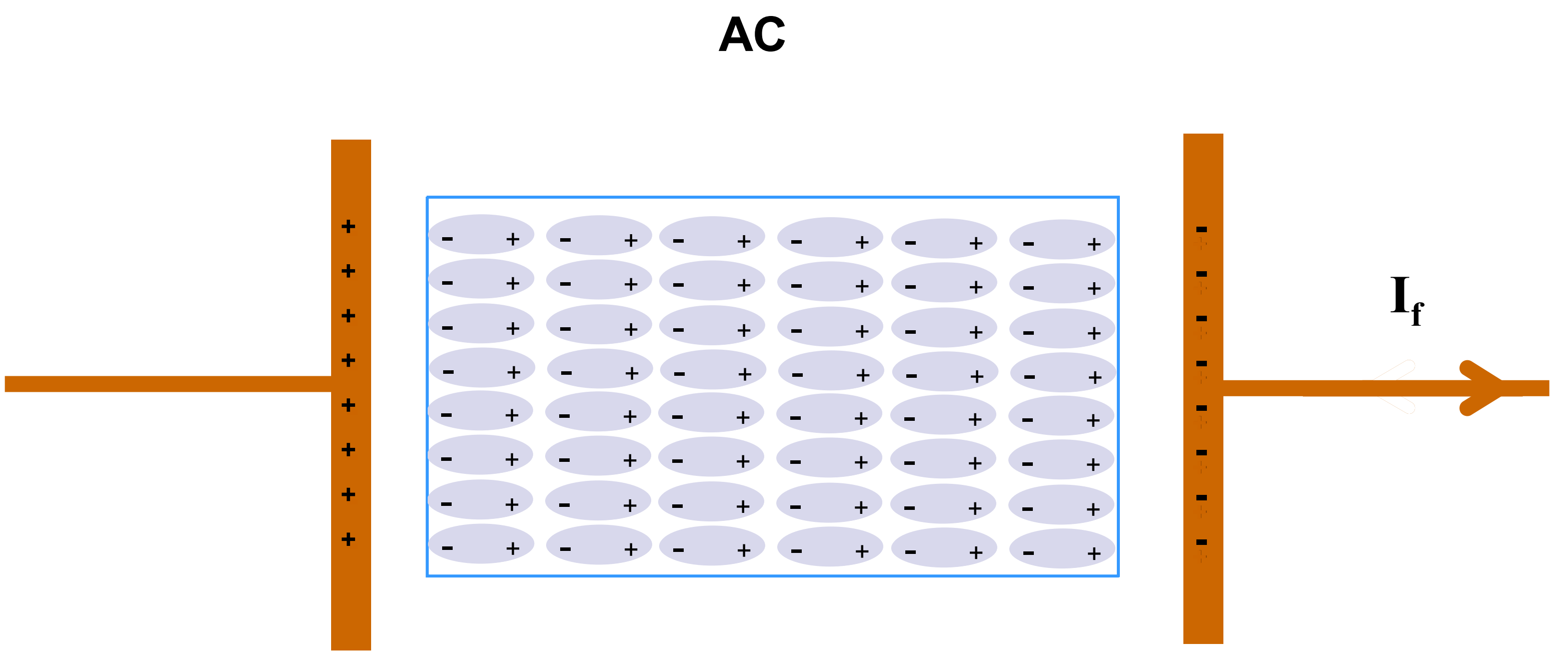
polarization current density

Polarization current



No dc current because dP/dt = 0
No "free" charge flow from one side to the other. $J_f = 0$.
So $J_t = 0$.
Note: D is the uniform between the plates.
E is different. E is not necessary // to D inside dielectric!!! (ε is a tensor!!)
For ac, oscillating dipole looks like the charge is crossing the dielectric !!

Polarization current



In magnetic materials

linear, homogeneous, isotropic

$$\vec{B} = \mu\vec{H} = \mu_o\mu_r\vec{H} = \mu_o(\vec{H} + \vec{M})$$

$$\nabla \times \vec{B} = \mu_o(\nabla \times \vec{H} + \nabla \times \vec{M})$$

$$\vec{J}_t = \vec{J}_f + \vec{J}_m = \mu_r\vec{J}_f$$

$$\vec{J}_m \equiv \nabla \times \vec{M}$$

is the equivalent current density of magnetized matter

$$\vec{M} = \chi_m\vec{H}$$

χ_m is the magnetic susceptibility

> 0	paramagnetic	quantum, spin, orbital
< 0	diamagnetic	free e^- (screening)
>>1	ferromagnetic	magnet
<< -1	anti-ferromagnetic	……etc

$$\vec{B} = \mu_o(\vec{H} + \chi_m\vec{H}) = \mu_o(1 + \chi_m)\vec{H}$$

$$\mu_r = 1 + \chi_m$$

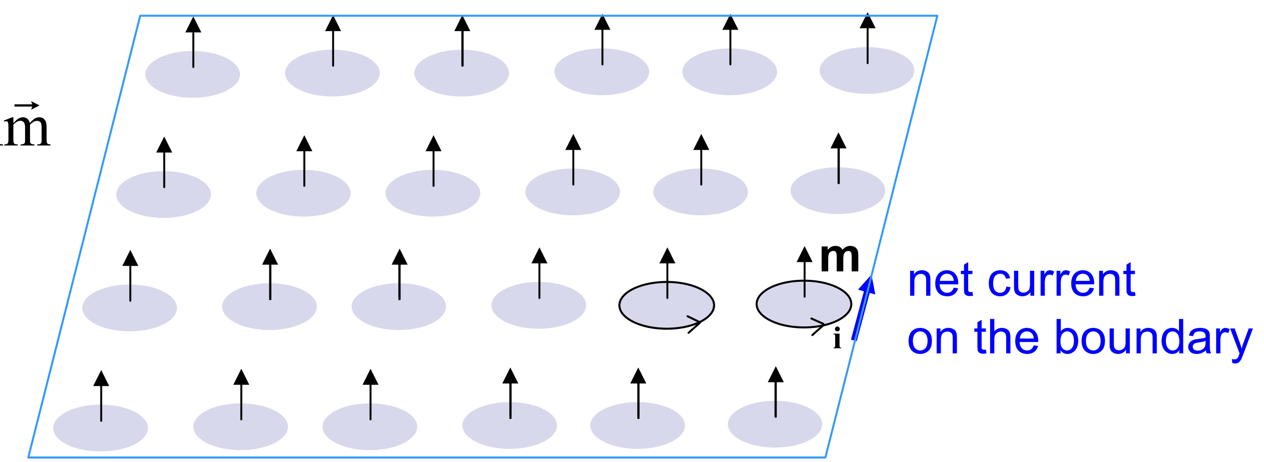
μ_r can be < 1 or negative !!!

note: J_m exists even in static field, but require spatial variation of M

Magnetization Current

e.g. in a perfectly magnetized 2D material, m = uniform inside and = 0 outside.

$$\vec{M} = n\vec{m}$$



In reality, m don't line up perfectly and has a spatial variation, which produce a volume magnetization current or bound current.

Again, no free electron transport from one place to another in macroscopic sense.

Total current density

$$\vec{J}_t = \vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M}$$

$$\boxed{\nabla \times \vec{B} = \mu_o\left(\vec{J}_t + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right)} = \mu_o\left(\vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M} + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

$$\boxed{\nabla \times \vec{B} = \mu_o\left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t} + \nabla \times \vec{M}\right)}$$

Check with Maxwell's

$$\nabla \cdot \nabla \times \vec{B} = 0 = \mu_o\left(\nabla \cdot \vec{J}_f + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{D}\right) + \nabla \cdot \nabla \times \vec{M}\right)$$

math

$$\nabla \cdot \nabla \times \vec{M} = 0$$ math

$$\nabla \cdot \vec{D} = \rho_f$$

$$\nabla \cdot \vec{J}_f + \frac{\partial \rho_f}{\partial t} = 0$$ continuity

OK

In terms of free charge and current

$$\nabla \times \vec{B} = \mu_o \left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t} + \nabla \times \vec{M} \right)$$

$$\vec{B} = \mu_o \left(\vec{H} + \vec{M} \right)$$

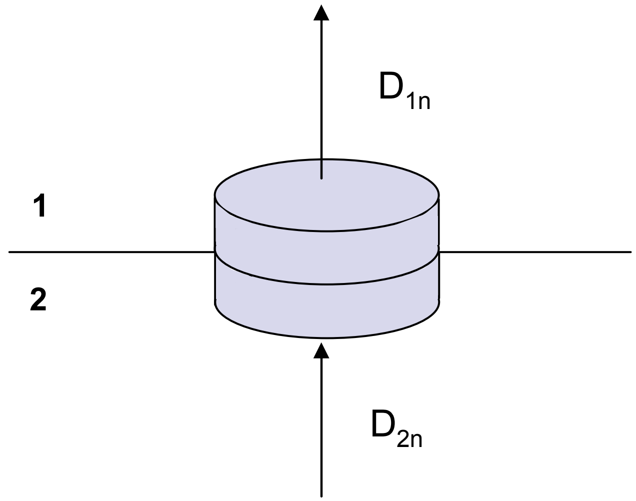
$$\left\{ \begin{array}{c} \nabla \cdot \vec{D} = \rho_f \\ \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{H} = \vec{J}_f + \dfrac{\partial \vec{D}}{\partial t} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \oint \vec{D} \cdot d\vec{a} = Q_f \\ \oint \vec{E} \cdot d\vec{\ell} = -\dfrac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a} \\ \oint \vec{B} \cdot d\vec{a} = 0 \\ \oint \vec{H} \cdot d\vec{\ell} = I_f + \dfrac{\partial}{\partial t} \int \vec{D} \cdot d\vec{a} \end{array} \right\}$$

no J_m

E & H vs. D & B

- H is related to free current
- B depends on material & history (hysteresis)
- H is measured when one builds electromagnets
- D is related to free charges
- But we measured voltage which is $E \cdot dl$ not charges
- Since E & H are what we measure directly, they are more commonly used than D & B.

Boundary Conditions - E

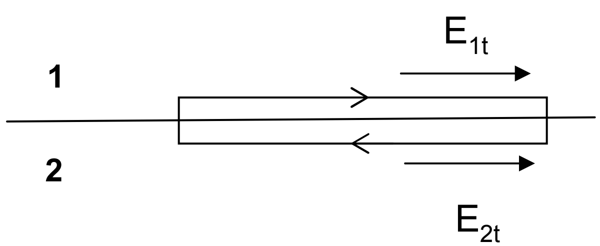


$$\oint \vec{D}\cdot d\vec{a} = Q_f$$

$$D_{1n}A - D_{2n}A = \sigma_s A$$

$$\boxed{D_{1n} - D_{2n} = \sigma_s}$$

surface charge density



$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a} = 0$$

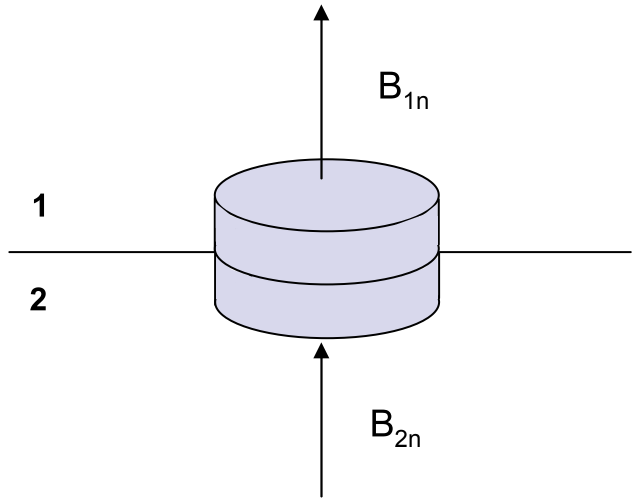
(area $\rightarrow 0$)

$$E_{1t}\ell - E_{2t}\ell = 0$$

$$\boxed{E_{1t} = E_{2t}}$$

same boundary conditions as in static case

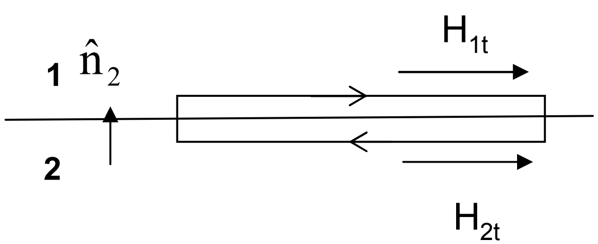
Boundary Conditions - H



$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$B_{1n}A - B_{2n}A = 0$$

$$\boxed{B_{1n} = B_{2n}}$$



$$\oint \vec{H} \cdot d\vec{\ell} = I_f - \frac{\partial}{\partial t}\int \vec{D} \cdot d\vec{a}$$

$$H_{1t}\ell - H_{2t}\ell = J_s\ell$$

$$\hat{n}_2 \times \left(\vec{H}_{1t} - \vec{H}_{2t}\right) = \vec{J}_s$$

$$\boxed{\hat{n}_2 \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_s}$$

same boundary conditions as in static case

Note: J_s only exists on conductors

Summary

Boundary conditions for the electric and magnetic fields.

Field Components	General Form	Medium 1 Dielectric / Medium 2 Dielectric	Medium 1 Dielectric / Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$	$E_{1t} = E_{2t} = 0$
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$ $D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$	$H_{1t} = J_s$ $H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$	$B_{1n} = B_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.

Phasor Equations

One can write E, H, or D, B in terms of time harmonic fields in form of: (Fourier)

$$\vec{E}(\vec{r},t) = \Re e\left\{\vec{E}(\vec{r})e^{j\omega t}\right\}$$

The amplitude E(r) is a Vector Phasor.
The Maxwell Equations (for E & H) and the corresponding phasor form are:

$$\left\{\begin{matrix} \varepsilon\nabla\cdot\vec{E} = \rho_f \\ \nabla\times\vec{E} = -\mu\dfrac{\partial\vec{H}}{\partial t} \\ \nabla\cdot\vec{H} = 0 \\ \nabla\times\vec{H} = \vec{J}_f + \varepsilon\dfrac{\partial\vec{E}}{\partial t} \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} \nabla\cdot\vec{E} = \dfrac{\rho_f}{\varepsilon} \\ \nabla\times\vec{E} = -j\omega\mu\vec{H} \\ \nabla\cdot\vec{H} = 0 \\ \nabla\times\vec{H} = \vec{J}_f + j\omega\varepsilon\vec{E} \end{matrix}\right\}$$

Exercise

$\vec{H}(\vec{r}, t) = \hat{x}0.01\cos(900t + \beta z)$ A/m in vacuum with no current source.

What is f, λ, **E** ?

$\omega = 2\pi f = 900$, $f = 143$ Hz

$\lambda = c/f = (3 \times 10^8)/(143) = 2094$ km

propagate to the –z direction

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

$\mathbf{E}(\mathbf{r}) = \mathbf{a}_y\, j3.77 \sin(900t + 3(10^{-6})z)$ V/m

$\mathbf{E}(z) = \mathbf{a}_y\, 3.77 \cos(900t + 3(10^{-6})z)$ V/m

Homework

1. Given that $\vec{E}(\vec{r},t) = \hat{y}0.1\sin(10\pi x)\cos(6\pi\cdot 10^9 t - \beta z)$ V/m in air, find **H**(**r**,t) and β using the phasor equations.

2. An infinite current sheet **J** = $\hat{x}5$ A/m coinciding with the xy-plane separates air (region 1, z > 0) from the medium with μ_{r2} = 2 (region 2, z < 0). Given that $\vec{H}_1 = 30\hat{x} + 40\hat{y} + 20\hat{z}$ A/m, find
   (a) $\mathbf{H}_2$,
   (b) $\mathbf{B}_2$,
   (c) angle α_1 that B_1 makes with the z-axis, and
   (d) angle α_2 that B_2 makes with the z-axis.

3. A 60 MHz electromagnetic wave exists in an air-dielectric coaxial cable having an inner conductor with radius a and an outer conductor with inner radius b. Assuming perfect conductors, and the phasor form of the electric field intensity to be (a < r < b) $\vec{E} = \hat{r}\dfrac{E_o}{r}e^{-jkz}$ V/m,
   (a) find k,
   (b) find **H** from the $\nabla \times \vec{E} = -j\omega\mu\vec{H}$,
   (c) find the surface current densities on the inner and outer conductors.