San José State University

Math 263: Stochastic Processes

# Brownian motion

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This lecture is based on the following textbook sections:

- Chapter 10 (Sections 10.1 - 10.3, 10.5)

**Outline of the presentation**

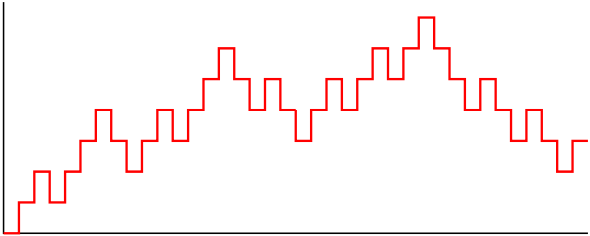
- Brownian motion: definitions and concepts
- Variations of Brownian motion

HW8

Consider the symmetric random walk over the set of integers

$$p_{i,i-1} = \frac{1}{2} = p_{i,i+1}, \quad i \in \mathbb{Z}$$

but we are going to speed up this process by taking smaller and smaller steps in smaller and smaller time intervals. This will converge to the Brownian motion process.



More precisely, suppose that we start off from $0$ and for each $\Delta t$ time unit we take a step of size $\Delta x$ either to the left or the right with equal probabilities.

If we let $X(t)$ denote the position at time $t$, then

$$X(t) = (X_1 + \cdots + X_n)\Delta x,$$

where $n = t/\Delta t$ and $X_1, X_2, \ldots$ are iid according to the following distribution

$$P(X_i = -1) = P(X_i = 1) = \frac{1}{2}$$

Since $\mathrm{E}(X_i) = 0,\ \mathrm{Var}(X_i) = 1$, we have

$$\mathrm{E}(X(t)) = 0, \quad \mathrm{Var}(X(t)) = n(\Delta x)^2 = \frac{(\Delta x)^2}{\Delta t} t$$

If we let $\Delta t, \Delta x \to 0$ but fix $\Delta x = \sigma \sqrt{\Delta t}$ for some positive constant $\sigma$, then

$$\mathrm{E}(X(t)) = 0, \quad \mathrm{Var}(X(t)) \to \sigma^2 t$$

A few observations about the limiting distribution:

- By the central limit theorem, $X(t) \sim N(0, \sigma^2 t)$.
- $\{X(t), t \geq 0\}$ has independent increments, that is, for all $0 = t_0 < t_1 < t_2 < \cdots < t_n$,

$$X(t_i) - X(t_{i-1}),\ \ i = 1, 2, \ldots, n$$

  are independent.
- $\{X(t), t \geq 0\}$ has stationary increments, i.e., the distribution of $X(t + s) - X(s)$ does not depend on $s$.
- $X(t)$ should be a continuous function of $t$.

**Def 0.1.** A stochastic process $\{X(t), t \geq 0\}$ is said to be a Brownian motion process, or a Wiener process, if

- $X(0) = 0$
- $\{X(t), t \geq 0\}$ has independent and stationary increments
- For every $t > 0$, $X(t) \sim N(0, \sigma^2 t)$

Remark. When $\sigma = 1$, the process is called standard Brownian motion.

Because any Brownian motion can be converted to the standard process by letting $B(t) = X(t)/\sigma$ we shall, unless otherwise stated, we will suppose that $\sigma = 1$:

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

Remark. $X(t)$ as a function on $[0, \infty)$ is continuous, but non-differentiable everywhere.

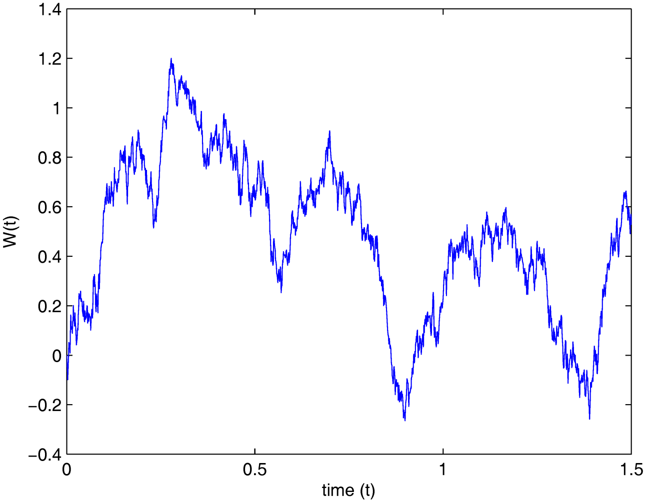
An informal proof is the following: Let $h > 0$ be a small number. Then

$$X(t+h) - X(t) \sim N(0, h).$$

and

$$\frac{1}{h}(X(t+h) - X(t)) \sim N(0, \frac{1}{h}).$$

As $h \to 0$, $X(t+h) - X(t) \to 0$ but $\frac{1}{h}(X(t+h) - X(t))$ does not converge.


1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
W(t)
0
0.5
1
1.5
time (t)

*Theorem* 0.1. The joint density function of $X(t_1), X(t_2), \ldots, X(t_n)$ for any $0 = t_0 < t_1 < t_2 < \cdots < t_n$ is

$$f(x_1, x_2, \ldots, x_n) = (2\pi)^{-n/2} \prod_{i=1}^{n} (t_i - t_{i-1})^{-1/2} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}\right)$$

*Proof.* Because

$$\begin{array}{lcr}
(X(t_0) = 0 = x_0) & & \\
X(t_1) = x_1 & \longrightarrow & X(t_1) - X(t_0) = x_1 - x_0 \\
X(t_2) = x_2 & \longrightarrow & X(t_2) - X(t_1) = x_2 - x_1 \\
\vdots & & \vdots \\
X(t_n) = x_n & \longrightarrow & X(t_n) - X(t_{n-1}) = x_n - x_{n-1}
\end{array}$$

we have

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{t_i - t_{i-1}}(x_i - x_{i-1})$$
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)$$

*Corollary* 0.2. The conditional density of $X(s)$ given $X(t) = B$ for $s < t$ is

$$f_{s|t}(x \mid B) = \frac{1}{\sqrt{2\pi s(t-s)/t}} e^{-\frac{(x-Bs/t)^2}{2s(t-s)/t}}$$

This implies that

$$\mathrm{E}(X(s) \mid X(t) = B) = Bs/t, \quad \mathrm{Var}(X(s) \mid X(t) = B) = s(t-s)/t$$

*Proof.*

$$
\begin{aligned}
f_{X(s)|X(t)}(x \mid B) = \frac{f_s(x) f_{t-s}(B-x)}{f_t(B)} &\propto e^{-x^2/2s} \cdot e^{-(B-x)^2/2(t-s)} \\
&\propto \exp\left(-\frac{1}{2}\left(\frac{t}{s(t-s)}x^2 - 2\frac{B}{t-s}x\right)\right) \\
&\propto \exp\left(-\frac{1}{2s(t-s)/t}\left(x - \frac{Bs}{t}\right)^2\right)
\end{aligned}
$$

Let $T_a$ denote the first time the Brownian motion process hits $a > 0$. Then one can prove the following result.

*Theorem* 0.3.

$$P(T_a < t) = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} e^{-y^2/2}\,\mathrm{d}y$$

*Proof.*

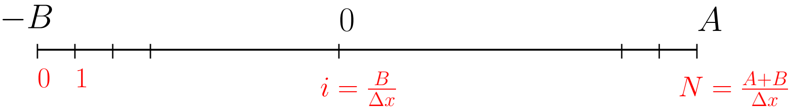
$$\begin{aligned}P(T_a < t) &= P(T_a < t, X(t) > a) + P(T_a < t, X(t) < a)\\ &= 2P(T_a < t, X(t) > a) = 2P(X(t) > a)\\ &= \frac{2}{\sqrt{2\pi t}} \int_a^{\infty} e^{-x^2/2t}\,\mathrm{d}x\end{aligned}$$

Remark. For $a<0$, $T_a = T_{-a}$ due to symmetry.

Remark. Another random variable of interest is the maximum value the process attains in $[0,t]$. Its distribution is obtained as follows:

$$P\left(\max_{0\le s\le t} X(s) \ge a\right) = P(T_a \le t), \quad a>0.$$

Let us show that the probability that Brownian motion hits $A$ before $-B$ where $A > 0, B > 0$ is $\frac{B}{A+B}$. To compute this we shall make use of the interpretation of Brownian motion as being a limit of the symmetric random walk.




By the results of the gambler's ruin problem,

$$P(\text{up } A \text{ before down } B) = \frac{B/\Delta x}{(A+B)/\Delta x} = \frac{B}{A+B}.$$

**Def 0.2.** We say that $\{X(t), t \geq 0\}$ is a Brownian motion process with drift coefficient $\mu$ and variance parameter $\sigma^2$ if

- $X(0) = 0$;
- $\{X(t), t \geq 0\}$ has stationary and independent increments;
- $X(t) \sim N(\mu t, \sigma^2 t)$ for all $t > 0$.

Remark. An equivalent definition is to let $\{B(t), t \geq 0\}$ be standard Brownian motion and then define

$$X(t) = \sigma B(t) + \mu t$$

**Def 0.3.** A stochastic process $\{Y(t), t \geq 0\}$ is called geometric Brownian motion if its logarithmic is a Brownian motion process (with drift coefficient $\mu$ and variance parameter $\sigma^2$):

$$\log X(t) = \underbrace{\sigma B(t) + \mu t}_{Y(t)} \quad \longrightarrow \quad X(t) = e^{Y(t)}$$

Geometric Brownian motion is useful in the modeling of stock prices over time when you feel that the percentage changes are independent and identically distributed.

For instance, suppose that $X_n$ is the price of some stock at time $n$. Then, it might be reasonable to suppose that $Y_n = X_n/X_{n-1}, n \geq 1$ are independent and identically distributed. It follows that

$$X_n = Y_n X_{n-1} = Y_n Y_{n-1} X_{n-2} = \cdots = Y_n Y_{n-1} \cdots Y_1 X_0$$

Therefore,

$$\log(X_n) = \sum_{i=1}^{n} \log(Y_i) + \log(X_0)$$

Since $\log(Y_i), i \geq 1$ are independent and identically distributed, $\{\log(X_n)\}$ will, when suitably normalized, approximately be Brownian motion with a drift, and so $\{X_n\}$ will be approximately geometric Brownian motion.

*Theorem* 0.4. For a geometric Brownian motion process $\{X(t), t \geq 0\}$,

$$\mathrm{E}[X(t) \mid X(u),\ 0 \leq u \leq s] = X(s)e^{(t-s)(\mu+\sigma^2/2)}, \quad s < t$$

*Proof.*

$$\begin{aligned}
\mathrm{E}[X(t) \mid X(u),\ 0 \leq u \leq s] &= \mathrm{E}[e^{Y(t)} \mid Y(u),\ 0 \leq u \leq s] \\
&= e^{Y(s)}\mathrm{E}[e^{Y(t)-Y(s)} \mid Y(u),\ 0 \leq u \leq s] \\
&= e^{Y(s)}\mathrm{E}[e^{Y(t)-Y(s)}] \\
&= e^{Y(s)}e^{\mu(t-s)+(t-s)\sigma^2/2}
\end{aligned}$$

where we have used the MGF $M_{N(\mu,\sigma^2)}(a) = e^{a\mu+a^2\sigma^2/2}$ at $a = 1$. □

Let $X(s), s \geq 0$ be a Brownian motion process with drift coefficient $\mu$ and variance parameter $\sigma^2$. Define

$$M(t) = \max_{0 \leq s \leq t} X(s)$$

We would like to determine the distribution of $M(t)$, the maximum of the process up to time $t$.

Remark. Let $T_y$ denote the first time the Brownian motion hits $y$. Then

$$T_y \leq t \qquad \longleftrightarrow \qquad M(t) \geq y.$$

Below is the main result about $M(t)$.

*Theorem* 0.5. Let $X(s), s \geq 0$ be a Brownian motion process with drift coefficient $\mu$ and variance parameter $\sigma^2$. Then

$$P(M(t) \geq y \mid X(t) = x) = e^{-2y(y-x)/t\sigma^2}, \quad y \geq x$$

This implies that

$$P(M(t) \geq y) = e^{2y\mu/\sigma^2}\bar{\Phi}\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right) + \bar{\Phi}\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right)$$

where $\bar{\Phi}(x) = 1 - \Phi(x)$ represents the complementary cdf of $N(0,1)$.

Outline of the proof:

To prove the theorem, we first need to derive the following lemma.

*Lemma*. For any $Y_1, \dots, Y_n \stackrel{\text{iid}}{\sim} N(\theta, v^2)$, the conditional distribution of $Y_1, \dots, Y_n$ given $\sum Y_i = x$ does not depend on $\theta$.

This result indicates that the sample total (or mean) from a normal population is a **sufficient statistic** for the population mean. That is, given the value of the statistic $(\sum Y_i)$, the sample provides no additional information about the target population parameter $(\theta)$.

A direct application of the lemma to the Brownian motion process with drift coefficient $\mu$ and variance parameter $\sigma^2$ shows that the conditional distribution of $X(s), 0 \leq s < t$ does not depend on $\mu$.

Specifically, fix $n$ and set $t_i = \frac{i}{n} t, i = 0, 1, \ldots, n$. We show that the conditional distribution of $X(t_1), \ldots, X(t_n), 0 = t_0 < t_1 < \ldots < t_n = t$ given $X(t) = x$ does not depend on $\mu$. To see this, let

$$Y_i = X(t_i) - X(t_{i-1}), \quad i = 1, \ldots, n$$

Then $Y_1, \ldots, Y_n$ are iid $N(\mu t/n, \sigma^2 t/n)$. By the lemma, the conditional distribution of $Y_1, \ldots, Y_n$ given $\sum_{i=1} Y_n = X(t) = x$ does not depend on $\mu$. It follows that the conditional distribution of $X(t_1), \ldots, X(n)$ given $X(t) = x$ does not depend on $\mu$.

We now derive the formula for the conditional distribution of $M(t)$ given $X(t) = x$. Note that the conditional distribution of $M(t)$ given $X(t) = x$ does not depend on $\mu$. Thus, without loss of generality, suppose $\mu = 0$.

We then consider the event that

$$X(t) \approx x, \qquad \text{i.e.,} \quad x \le X(t) \le x + h$$

for small $h$. The probability is

$$P(X(t) \approx x) \approx f_{X(t)}(x) \cdot h$$

We have

$$
\begin{aligned}
P(M(t) \geq y \mid X(t) \approx x) &= \frac{P(M(t) \geq y, X(t) \approx x)}{P(X(t) \approx x)} \\
&= \frac{P(T_y \leq t, X(t) \approx x)}{P(X(t) \approx x)} \\
&= \frac{P(T_y \leq t, X(t) \approx 2y - x)}{P(X(t) \approx x)} \\
&= \frac{P(X(t) \approx 2y - x)}{P(X(t) \approx x)} \\
&= \frac{f_{X(t)(2y-x)}}{f_{X(t)(x)}} \\
&= e^{-2y(y-x)/t\sigma^2}.
\end{aligned}
$$

Lastly, to derive the formula for $P(M(t) \geq y)$, we condition on $X(t) = x$:

$$\begin{aligned}
P(M(t) \geq y) &= \int_{-\infty}^{\infty} P(M(t) \geq y \mid X(t) = x) f_{X(t)}(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{y} P(M(t) \geq y \mid X(t) = x) f_{X(t)}(x)\,\mathrm{d}x + \int_{y}^{\infty} 1 \cdot f_{X(t)}(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{y} e^{-2y(y-x)/t\sigma^2} \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-(x-\mu t)^2/2t\sigma^2}\,\mathrm{d}x + P(X(t) > y)\,\mathrm{d}x \\
&= e^{2\mu y/\sigma^2} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-(x-\mu t-2y)^2/2t\sigma^2}\,\mathrm{d}x + P(X(t) > y) \\
&= e^{2\mu y/\sigma^2} \Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right) + \bar{\Phi}\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right)
\end{aligned}$$